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ON TWO CLASSES OF SETS CONTAINING ALL BAIRE SETS AND ALL CO-ANALYTIC SETS

Zoltan BALOGH
Kossuth University, Debrecen, H-4010, Hungary

Harold BENNETT
Department of Mathematics, Texas Tech University, Lubbock, TX 79409, USA

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The notions of an S_δ set and an R_δ set are introduced A space is S-perfect (R-perfect) if each closed set is an S_δ set (R_δ set) Conditions are given which indicate when spaces are S-perfect or R-perfect Examples are given of spaces which do not have these properties and examples are given of spaces with these properties that are not perfect spaces The images of S-perfect and R-perfect spaces under various types of mappings are investigated

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Introduction

As is well known G_δ sets and F_σ sets play a central role in general topology On the other hand, sets at higher levels of the descriptive hierarchy, although intensively studied on their own, rarely replace F_σ sets or G_δ sets in typical theorems of general topology such as metrizability theorems, extension theorems, etc. (some examples of results of this kind can be found in [15, 8, 4, 13, 14]).

The idea of this paper arose when the authors tried to prove some topological theorems involving sets at higher levels of the descriptive hierarchy in topological spaces, such as Baire sets, open-analytic sets or co-analytic sets. It usually turned out that the hypothesis really needed in those theorems involved what the authors call S_δ sets and R_δ sets in this paper. Thus they began to study S_δ sets and R_δ sets for their own sake. This paper is devoted to that study and applications will only be given for Baire sets The reader, however, is invited to reformulate some of our theorems by substituting "co-analytic" or "open-analytic" in place of "S_δ" and "Baire" in place of "R_δ".

The material of the paper is organized in four sections In Section 1, S_δ sets and R_δ sets are defined. A consequence of this definition is that all subsets of, say, the real line are both S_δ sets and R_δ sets. Thus such sets may only bear some significance in the context of more general topological spaces The aim of this paper is to show that in that context they really have some significance. Also in Section 1, partial answers are given to the question when all subsets of a metrizable space are S_δ sets

In Section 2, S-perfect spaces and R-perfect spaces are introduced by replacing G_δ by S_δ and R_δ, respectively, in the definition of perfect spaces. After listing the basic properties of such spaces it will be shown that not all metacompact Moore spaces are R-perfect, giving rise to the question of what is the effect of R-perfectness on Moore spaces It is also shown that quasi-developable spaces may fail to be S-perfect.

Section 3 contains a number of (sometimes consistency) results concerning conditions which imply that certain S_δ sets or R_δ sets are G_δ sets Some applications are given concerning a problem of Ross and Stromberg asking whether closed Baire sets are zero-sets in normal T_0, locally compact spaces

Section 4 gives a complete analysis of the preservation of S-perfectness and R-perfectness of topological spaces under various types of mappings.

For some applications of S-perfectness and R-perfectness in metrizability theory of manifolds, see [3]

In this paper space will mean topological space Our terminology and notation will follow the standards of set theory and set-theoretic topology as is used in [17, 18]

Some of the definitions of well-known topological concepts will be recalled here for the reader's convenience. Given a space X, a subset A of X is said to be

(a) *analytic*, if A can be obtained by one application of the Souslin operation from a countable family of closed subsets of X, i e, A is a set of the form $\bigcap\{\bigcup\{F_{\psi|n}\ n\in\omega\}: \psi\in{}^{\omega}\omega\}$, where all the $F_{\psi|n}$'s are closed sets,

(b) *co-analytic*, if $X-A$ is analytic,

(c) *open-analytic*, if A can be obtained by one application of the Souslin operation from a countable family of open subsets of X,

(d) *Baire*, if A belongs to the σ-algebra generated by the zero-sets of X

A subset A is said to be a *regular* G_δ set, if there is a countable family $\mathcal{G}$ of open subsets of X with $\bigcap\mathcal{G}=\bigcap\{\mathrm{cl}(G)\ G\in\mathcal{G}\}=A$ A space is said to have a *regular G_δ-diagonal*, if the diagonal Δ is a regular G_δ set in the product space $X\times X$.

A space X is said to be *quasi-developable* [5] if there is a sequence $\{\mathcal{G}_n .\ n\in\omega\}$ of families of open subsets of X such that for each $x\in X$, $\{\mathrm{st}(x,\mathcal{G}_n)\ n\in\omega\}$ is a neighborhood base for X If it also is required that each $\mathcal{G}_n$ covers X, then X is said to be developable Regular T_1, developable spaces are called Moore spaces.

A cover $\mathcal{G}$ of a space X is said to be *point separating* (respectively *strongly separating*) if for every pair of distinct points $x, y\in X$, there is a $G\in\mathcal{G}$ with $x\in G$,

$y \notin G$ (respectively $x \in G$, $y \notin \mathrm{cl}(G)$). Other concepts will be defined at the place of their first appearance.

1. S_δ and R_δ sets in topological spaces

Definition 1.1. Let A be a subset of a topological space X and let $\mathcal{G}$ be a family of subsets in X $\mathcal{G}$ is said to *separate* (respectively *strongly separate*) the points of A from the points of $X-A$ if for every $x \in A$ and $y \in X-A$, there is a $G \in \mathcal{G}$ with $x \in G$, $y \notin G$ (respectively with $x \in G$, $y \notin \mathrm{cl}(G)$ where $\mathrm{cl}(G)$ denotes the closure of G)

Definition 1.2. A subset A of a topological space X is said to be an S_δ (respectively an R_δ) *set* in X if there is a countable family $\mathcal{G}$ of open subsets of X separating (respectively strongly separating) the points of A from the points of $X-A$.

Let $\mathcal{S}(X)$ (respectively $\mathcal{R}(X)$) denote the set of all S_δ (respectively R_δ) subsets of X. The following two propositions list some of the basic properties of $\mathcal{S}(X)$ and $\mathcal{R}(X)$.

Theorem 1.1. *Let X be a topological space. Then $\mathcal{S}(X)$ possesses the following properties*:

(a) *$\mathcal{S}(X)$ is closed under countable unions and countable intersections*

(b) *Every G_δ subset of X is an S_δ set in X.*

(c) *Every co-analytic and every open-analytic subset of X is an S_δ set in X*

(d) *If X has a countable point separating open cover, then every subset of X is an S_δ set.*

Proof. All of the above properties directly follow from the definitions. □

Theorem 1.2. *Let X be a topological space Then $\mathcal{R}(X)$ possesses the following properties*:

(a) *$\mathcal{R}(X)$ forms a σ-algebra.*

(b) $\mathcal{R}(X) \subseteq \mathcal{S}(X)$.

(c) *Every zero-set in X is an R_δ set.*

(d) *Every Baire set in X is an R_δ set.*

(e) *If X has a countable strongly separating open cover, then every subset of X is an R_δ set*

Proof. (a) To see that $\mathcal{R}(X)$ is closed under countable unions, observe that if for each $n \in \omega$, $\mathcal{G}_n$ strongly separates the points of A_n from the points of $X-A_n$, then $\mathcal{G} = \bigcup\{\mathcal{G}_n : n \in \omega\}$ strongly separates the points of $A = \bigcup\{A_n : n \in \omega\}$ from the points of $X-A$ To see that $\mathcal{R}(X)$ is closed under taking complements note that if $\mathcal{G}$

strongly separates the points of A from the points of $X-A$, then $\{X-\mathrm{cl}(G)\ G\in\mathscr{G}\}$ strongly separates the points of $X-A$ from the points of $X-(X-A)=A$

(b), (c) and (e) directly follow from the definitions. Finally, (d) follows from (c) and (a) □

Remarks. (1) The following example shows that $\mathscr{S}(X)$ may fail to be a σ-algebra. Let $X=D\cup\{y\}$ be the one-point compactification of an uncountable discrete space D Then $D\in\mathscr{S}(X)$, but $X-D=\{y\}\notin\mathscr{S}(X)$

(2) The following types of space have a countable separating open cover and thus satisfy the conclusion of Theorem 1.1(d)

(A) metrizable spaces of cardinality $\leqslant 2^{\omega}$ (see [21]),

(B) More spaces of cardinality $\leqslant 2^{\omega}$ (see [21]),

(C) Lindelof spaces with a G_δ diagonal

(3) The following types of spaces have a countable strongly separating open cover and thus satisfy the conclusion of Theorem 1 2(e)

(A′) metrizable spaces of cardinality $\leqslant 2^{\omega}$ (see [21]),

(B′) normal Moore spaces of cardinality $\leqslant 2^{\omega}$ (see [21]),

(C′) Lindelof spaces with a regular G_δ diagonal

By Remark (1) and Theorem 1.2(a) above we may have $\mathscr{R}(X)$ properly contained in $\mathscr{S}(X)$ even for hereditarily paracompact, compact Hausdorff spaces X A natural problem is to find conditions under which S_δ sets are R_δ sets The following proposition gives several such conditions

Theorem 1.3. (a) *If X is a hereditarily normal T_2-space, and A is a closed S_δ subset of X, then A is an R_δ set in X*

(b) *If X is a perfectly normal T_2-space, then $\mathscr{R}(X)=\mathscr{S}(X)$.*

Proof. (a) By taking complements it follows that there is a countable family $\mathscr{F}$ of closed sets which separates the points of $X-A$ from the points of A Since X is a hereditarily normal space for each F in $\mathscr{F}$, there is a pair $G_1(F)$, $G_2(F)$ of disjoint open sets with $G_1(F)\supset F-A$, $G_2(F)\supset A-F$ Then it is easy to see that $\mathscr{G}=\{G_i(F)\colon F\in\mathscr{F},\ i=1,2\}$ strongly separates the points of A from the points of $X-A$

(b) Only $\mathscr{S}(X)\subseteq\mathscr{R}(X)$ has to be proven. So let A be an arbitrary S_δ subset of X, and $\mathscr{G}$ be a countable family of open sets which separates the points of A from the points $X-A$. By perfect normality, for each $G\in\mathscr{G}$ there is countable family $\mathscr{H}(G)$ of open sets with $\bigcup\mathscr{H}(G)=\bigcup\{\mathrm{cl}(H).\ H\in\mathscr{H}(G)\}=G$ Then $\mathscr{G}'=\bigcup\{\mathscr{H}(G)\colon G\in\mathscr{G}\}$ strongly separates the points of A from the points of $X-A$ □

By Remark (3) following Theorem 1.2, every subset of a metrizable space of cardinality $\leqslant 2^{\omega}$ is an R_δ subset We are going to see below that this is not the case for all metrizable spaces. In fact the following problem seems to be quite difficult and the solution is likely to need deep combinatorics.

Problem 1.1. Characterize all weakly Q-spaces, i e., all metrizable spaces each subset of which is an S_δ set

The following theorem shows that most metrizable spaces of cardinality $>2^\omega$ are not weakly Q-spaces

Theorem 1.4. *If X is a metrizable weakly Q-space, then $X = Y \cup Z$ in such a way that $Y \cap Z = \emptyset$ and*

(a) *Y is a closed subspace of weight $<2^{2^\omega}$,*

(b) *each separable subspace of Z has cardinality $<2^\omega$*

Under GCH or even under just $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ Theorem 1.4 turns into the following

Corollary 1.1 (GCH). *If X is a metrizable weakly Q-space, then $X = Y \cup Z$ in such a way that $Y \cap Z = \emptyset$ and*

(a) *Y is a closed subspace of weight and thus, of cardinality $\leq \omega_1$,*

(b) *each separable subspace of Z is countable.*

In order to prove Theorem 1.4 we need the following two lemmas.

Lemma 1.1. *Let $\lambda = 2^{2^\omega}$, H be a set of cardinality 2^ω, and $\langle H_\alpha : \alpha \in \lambda \rangle$ be a λ-sequence of subsets of H each of cardinality 2^ω. Then there is a λ-sequence p_α $H_\alpha \to 2$ of functions such that for every function $f : \lambda \to 2^\omega$ there are $\alpha, \beta \in \lambda$ such that $f(\alpha) = f(\beta)$ and $p_\alpha^{-1}(1) \cap p_\beta^{-1}(0) \neq \emptyset$*

Proof. Let $c = 2^\omega$ and let $\langle \varphi_\alpha \ \ \alpha \in \lambda \rangle$ enumerate all c-sequences of $\varphi = \langle f_\xi : \xi \in c \rangle$ of functions $f_\xi : H \to 2$ For every $\alpha \in \lambda$ define p_α $H_\alpha \to 2$ in such a way that if $\varphi_\alpha = \langle f_\xi \ \ \xi \in c \rangle$, then

$$\text{for each } \xi \in c \text{ there is an } x \in H_\alpha \text{ with } f_\xi(x) \neq p_\alpha(x). \tag{1}$$

Since H_α has cardinality c this can be done by induction of $\xi \in c$.

In order to prove that this choice of the p_α's is as required, take an arbitrary function $f : \lambda \to c$ and, for each $\xi \in c$, define $f_\xi . H \to 2$ by putting, for every $x \in H$ $f_\xi(x) = 0$ if there is an $\alpha \in \lambda$ such that $x \in H_\alpha$, $f(\alpha) = \xi$ and $p_\alpha(x) = 0$. Let $f_\xi(x) = 1$ otherwise.

Then there is an $\alpha \in \lambda$ with $\varphi_\alpha = \langle f_\xi . \xi \in c \rangle$. Let $f(\alpha) = \xi$ By (1) there is an $x \in H_\alpha$ with $f_\xi(x) \neq p_\alpha(x)$ We are going to show that there is a $\beta \in \lambda$ such that $f(\beta) = \xi$ and $x \in p_\alpha^{-1}(1) \cap p_\beta^{-1}(0)$ thereby finishing the proof of the lemma

To see this we first show that $f_\xi(x) = 0$. Indeed, since $f_\xi(x) \neq p_\alpha(x)$ from $f_\xi(x) = 1$ it would follow that $p_\alpha(x) = 0$. So the definition of $f_\xi(x)$ would imply that $f_\xi(x) = 0$, a contradiction

Thus $f_\xi(x)=0$. By $f_\xi(x)\neq p_\alpha(x)$ it follows that $p_\alpha(x)=1$. Furthermore, again by the definition of $f_\xi(x)$, there is a $\beta\in\lambda$ with $x\in H_\beta$, $f(\beta)=\xi$ and $p_\beta(x)=0$. Now $f(\alpha)=f(\beta)=\xi$, $x\in H_\alpha\cap H_\beta$, $p_\alpha(x)=1$ and $p_\beta(x)=0$ Since $f:\lambda\to c$ was arbitrary we are finished □

Lemma 1.2. *Let $\lambda=2^{2^\omega}$ be equipped with the discrete topology, let H be the Hilbert cube and let X be a subspace of the topological product $H\times\lambda$ such that for each $\alpha\in\lambda$, $X_\alpha=X\cap(H\times\{\alpha\})$ has cardinality 2^ω Then there is a subset $A\subset X$ such that A is not an S_δ subset in X*

Proof. Let $\pi.H\times\lambda\to H$ be the natural projection defined by $\pi(\langle x,\alpha\rangle)=x$ and let $H_\alpha=\pi(X_\alpha)$ for each $\alpha\in\lambda$ Note that $\langle x,\alpha\rangle\in X$ if and only if $x\in H_\alpha$. Consider a sequence p_α $H_\alpha\to 2$ $(\alpha\in\lambda)$ of functions such as described in Lemma 1.1 and define $A=\{\langle x,\alpha\rangle\in X: p_\alpha(x)=1\}$. Suppose indirectly that A is an S_δ subset in X, i.e., there is a countable family $\mathcal{G}$ of open subsets of $H\times\lambda$ in such a way that

$$\text{for every } a\in A \text{ and } b\in X-A \text{ there is a } G\in\mathcal{G} \text{ such that } a\in G \text{ and } b\notin G \tag{2}$$

For each $\alpha\in\lambda$, let g_α be the map from $\mathcal{G}$ into the family of open subsets of H defined by

$$g_\alpha(G)=\pi(G\cap(H\times\{\alpha\}))$$

for each $G\in\mathcal{G}$.

Note that for every $\langle x,\alpha\rangle\in H\times\lambda$ and $G\in\mathcal{G}$, $\langle x,\alpha\rangle\in G$ if and only if $x\in g_\alpha(G)$. Since $\mathcal{G}$ is countable and there are 2^ω open subsets of H, there are 2^ω maps from $\mathcal{G}$ into the family of open subsets of H Thus, by Lemma 1 1, there are $\alpha,\beta\in\lambda$ such that $g_\alpha=g_\beta$ and there is a point $x\in H_\alpha\cap H_\beta$ with $p_\alpha(x)=1$ and $p_\beta(x)=0$. Then $a=\langle x,\alpha\rangle\in A$ and $b=\langle x,\beta\rangle\in X-A$ On the other hand, for every $G\in\mathcal{G}$, $\langle x,\alpha\rangle\in G$ holds if and only if $x\in g_\alpha(G)=g_\beta(G)$ if and only if $\langle x,\beta\rangle\in G$ in contradiction with (2). □

Remark. By Lemma 1.2 whenever S is a separable metrizable space of cardinality 2^ω and $\lambda=2^{2^\omega}$ is equipped the discrete topology, then the topological product $S\times\lambda$ (i.e , the free sum of λ copies of S) is an example of a metrizable space that is not a weakly Q-space

In fact taking $\lambda=(2^\omega)^+$ is enough to guarantee that $S\times\lambda$ is not a weakly Q-space. This is so because in order to carry out the proof of Lemma 1 2 in this special case (when all X_α's are the same) instead of applying Lemma 1.1 it is enough simply to make sure that all the sets $p_\alpha^{-1}(1)$, $\alpha\in\lambda$, are distinct The difficult thing is to deal with distinct X_α's as was done in Lemmas 1.1 and 1.2.

Having proved the lemmas we are in a position to give a proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\mathcal{B}$ be a σ-discrete base for X and let $\mathcal{B}_1=\{B\in\mathcal{B}: B$ contains no separable subspaces of cardinality $2^\omega\}$. Let $Z=\bigcup\mathcal{B}_1$ and $Y=X-Z$ We need to show that this choice of Y and Z satisfy (a) and (b) respectively.

Suppose that (a) fails, i.e., Y has weight $\geq 2^{2^\omega}$ Then, since $\mathscr{B}$ is σ-discrete, there is a discrete family $\mathscr{B}_2 \subseteq \mathscr{B}$ of cardinality 2^{2^ω} such that $B \in \mathscr{B}_2$ implies $B \cap Y \neq \emptyset$ By definition of Y each $B \in \mathscr{B}_2$ contains a separable subspace $X(B)$ of cardinality 2^ω Then the subspace $X_1 = \bigcup \{X(B): B \in \mathscr{B}_2\}$ is the free union of 2^{2^ω} subspaces each homomorphic to a subspace of cardinality 2^ω of the Hilbert cube H. By Lemma 1.2, X_1 contains a subset which is not an S_δ subset of X_1 (a fortiori, of X) in contradiction with our assumption that X was a weakly Q-space.

Suppose now that (b) fails, i.e., Z contains a separable subspace Z_1 of cardinality 2^ω. Since $\mathscr{B}_1$ is σ-discrete, there are only countably many members of $\mathscr{B}_1$, which meet Z_1. Thus there is a $B \in \mathscr{B}_1$ such that $|B \cap Z_1| = 2^\omega$, in contradiction with the definition of $\mathscr{B}_1$. □

Remark. In order to facilitate the solution of Problem 1 1 another partial result will be mentioned. To formulate this result let us call a metrizable space X *standard* if X is the union of an increasing family $\{X_n^* \ n \in \omega\}$ of subspaces such that each X_n^* is the free sum of its clopen subspaces of cardinality $\leq 2^\omega$. It can easily be seen that a standard space X is a weakly Q-space if and only if each of the subspaces X_n^* is a weakly Q-space. Then the following result holds·

$$\text{Assuming } V = L, \text{ every weakly } Q\text{-space } X \text{ is standard.} \tag{3}$$

The sketch of the proof of (3) is as follows:

Note first that every S_δ subset of X is a G_{2^ω} set, i.e , it can be represented as the intersection of 2^ω open sets. Thus for X we can follow the proof of the main result in [4] to conclude that X is the union of a family $\{D_\xi : \xi < 2^\omega\}$ of closed discrete subspaces of X No generality is lost if it is assumed that the D_ξ's are pairwise disjoint.

Let $\mathscr{B} = \bigcup \{\mathscr{B}_n \ n \in \omega\}$ be a σ-discrete base for X. Fix $\xi < 2^\omega$ and for each $x \in D_\xi$ fix $B(x)$ in $\mathscr{B}$ such that $\{B(x): x \in D_\xi\}$ is a pairwise disjoint expansion of the points of D_ξ Let $X_n = \{x\cdot B(x) \in \mathscr{B}_n\}$. Notice that $\bigcup \mathscr{B}_n \supset X_n$. If $B \in \mathscr{B}_n$ and $x \in B \cap X_n$, then $B = B(x)$. Thus $|B \cap D_\xi \cap X_n| \leq 1$ for each $B \in \mathscr{B}_n$. Hence $|B \cap X_n| \leq 2^\omega$ for each $B \in \mathscr{B}_n$ Thus $\{B \cap X_n . B \in \mathscr{B}_n\}$ is a discrete family of sets of size $\leq 2^\omega$. For each x in X and $n \in \omega$ there is an open set V containing x such that $|V \cap \bigcup_{j \leq n} X_j| \leq 2^\omega$. Thus each $X_n^* = \bigcup_{j \leq n} X_j$ is a metrizable space of local cardinality $\leq 2^\omega$. By a standard chaining argument, X_n^* is the free union of spaces each of cardinality $\leq 2^\omega$. Hence X is standard.

There are two flaws of this result, however. First, it is unknown to the authors whether the axiom of constructability can be dropped. Second, although metrizable spaces of cardinality $\leq 2^\omega$ are weakly Q-spaces (see Remark (3), it is unknown to the authors when free sums of metrizable spaces of cardinality $\leq 2^\omega$ are weakly Q-spaces. Theorem 1.4 shows that not all such free sums are weakly Q-spaces. Thus the lengthy proof which makes use of the methods of [4] is omitted.

Another interesting problem is when S_δ (or R_δ) subsets of a space are G_δ sets This will be dealt with in Section 3

2. S-perfect and R-perfect spaces

Definition 2.1. A topological space X is said to be *S-perfect* (respectively *R-perfect*) if every closed subset of X is an S_δ (respectively R_δ) set in X.

The following two propositions follow from Theorems 1 1 and 1 2

Theorem 2.1. *The following classes of topological spaces are S-perfect:*
(i) *perfect spaces,*
(ii) *spaces in which closed sets are co-analytic sets,*
(iii) *spaces in which closed sets are open-analytic sets, and*
(iv) *spaces with a countable point separating open cover*

Theorem 2.2. *The following classes of topological spaces are R-perfect:*
(i) *perfectly normal spaces,*
(ii) *spaces in which closed sets are Baire sets,*
(iii) *spaces with a countable strongly separating open cover.*

All developable spaces are S-perfect since they are perfect On the other hand, even metacompact Moore spaces may fail to be R-perfect as we shall deduce from the following lemma. For a definition of a Pixley–Roy hyperspace and the terminology in the proof of the next lemma, see [23]

Lemma 2.1. *If X is a metrizable space and its Pixley–Roy space* $\mathrm{PR}(X)$ *is R-perfect, then every subset of X is an S_δ set in X*

Proof. Let A be an arbitrary subset of X. It is enough to show that there is a countable family $\mathscr{F}$ of subsets of X such that for every pair of points x and y such that $x \in A$, $y \in X - A$, there is an F in $\mathscr{F}$ with $x \in F$, $y \notin \mathrm{cl}_X(F)$

Indeed, the conclusion of the above sentence implies that $X - A$ is an S_δ subset of X. Since A was arbitrary it follows that all subsets of X are S_δ sets

To see that such a family $\mathscr{F}$ can be constructed, let $\mathscr{G}$ be a countable open family in $\mathrm{PR}(X)$ which strongly separates the points of $A^* = \{\{x\}: x \in A\}$ from the points of $\mathrm{PR}(X) - A^*$.

If $G \in \mathscr{G}$ and $\{x\} \in G$, let $B(x, G)$ be a ball around x, in the metric of X, such that $[\{x\}, B(x, G)] \subset G$ For every $n \in \omega$ and $G \in \mathscr{G}$ define

$$F(n, G) = \{x \in A: \{x\} \in G \text{ and the radius of } B(x, G) \text{ is } \geq 2^{-n}\}.$$

Clearly, $\{x \in A: \{x\} \in G\} = \bigcup \{F(n, G): n \in \omega\}$.

It follows that $\mathscr{F} = \{F(n, G): n \in \omega,\ G \in \mathscr{G}\}$ is as required.

Indeed, let $x \in A$, $y \in X - A$ be arbitrary. By the choice of $\mathscr{G}$, there is a $G \in \mathscr{G}$ with $\{x\} \in G$, $\{y\} \notin \mathrm{cl}_{\mathrm{PR}(X)}(G)$. Let n be so big that $x \in F(n, G)$. Since $\{y\} \notin \mathrm{cl}_{\mathrm{PR}(X)}(G)$, there is an open ball $V(y, G)$ of radius $\leqslant 2^{-n}$ around y, in the metric of X, such that $[\{y\}, V(y, G)] \cap G = \emptyset$. It follows that

$$V(y, G) \cap F(n, G) = \emptyset. \tag{4}$$

Indeed, if there was an element $z \in V(y, G) \cap F(n, G)$, then, since the radius of $V(y, G)$ was $\leqslant 2^{-n}$, and the radius of $B(z, G)$ was $\geqslant 2^{-n}$, it would follow that

$$\{y, z\} \subset V(y, G) \cap B(z, G).$$

Hence

$$\{y, z\} \in [\{y\}, V(y, G)] \cap [\{z\}, B(z, G)] \subset (X - G) \cap G = \emptyset$$

which is a contradiction

Thus, (4) holds. From (4), however, it follows that $y \notin \mathrm{cl}_X(F(n, G))$ Since $x \in F(n, G)$, this concludes the proof. □

Remark. Note that by Theorem 1.3(b), $\mathscr{S}(X) = \mathscr{R}(X)$ for metrizable spaces. Thus the conclusion of Lemma 2.1 may be strengthened to "every subset of X is an R_δ set in X".

Example 2.1. There is a metacompact Moore space which is not R-perfect

Proof. By Theorem 1 4 there is a metrizable space X which is not a weakly Q-space. Then PR(X) is a metacompact Moore space (cf. [23]) which cannot be R-perfect by Lemma 2.1. □

Remarks. (1) Several common examples of Moore spaces, such as the tangent disc space, have a countable strongly separating open cover and are, thus, R-perfect

(2) A developable, nonmetrizable manifold such as the Pruefer manifold (see [20]) gives another example of a Moore space which is not R-perfect Indeed, it is shown in [3] that R-perfect Moore manifolds are metrizable.

(3) Since not all Moore spaces are R-perfect the question naturally arises what the effect of R-perfectness is on collectionwise normality type properties of Moore spaces. There are several plausible conjectures the most apparent of which is the following problem.

Problem 2.1. It is true or consistent with the usual axioms of set theory that if X is an R-perfect Moore space, and A is a closed discrete subset of X, then A is the union of a disjoint family $\mathscr{A}$ of its subsets such that

(a) $A \in \mathscr{A}$ implies $|A| \leqslant 2^\omega$,

(b) $\mathscr{A}$ can be separated by disjoint open sets.

Another natural question is whether quasi-developable spaces [5] are S-perfect The following two examples give negative answers to this question in various nice classes of spaces.

Example 2.2. There is a hereditarily paracompact, quasi-developable T_2 space which is not S-perfect.

Proof. By Theorem 1 4 there is a metrizable space X with a subset A which is not S_δ in X Modify the topology of X in the following way.

(1) Points of A keep their original neighborhoods.

(2) Points of $X-A$ become isolated

Then X with this modified topology will satisfy the conditions in Example 2 2. □

Example 2.3. There is a locally compact, locally countable, quasi-developable T_2 space Z of cardinality $\leq 2^\omega$ which is not S-perfect.

Proof. The following space due to Isbell will serve our purpose. Let D be a set of cardinality w_1 and A be a maximal almost disjoint family of countable subsets of D.

The underlying set of the space Z will be $A \cup D$ The topology of Z is given by the following two conditions

(1) Each point of D is isolated

(2) A neighborhood base for $x \in A$ is given by $\{\{x\} \cup (x-F)$: F is a finite subset of $x\}$

It is easily seen that Z is locally compact, locally countable, quasi-developable and Hausdorff.

Suppose indirectly that Z is S-perfect Then there is a countable family $\mathscr{G}$ of open subsets of Z which separates the points of A from the points of $Z-A=D$

Claim. If $d \in D$, then there is a finite subset $\mathscr{G}(d)$ of $\{G \in \mathscr{G}: d \notin G\}$ such that $D-\bigcup \mathscr{G}(d)$ is finite

Indeed, since $\mathscr{G}$ separates the points of A from $d \in D$, $\{G \in \mathscr{G}. d \notin G\}$ covers A List the members of $\{G \in \mathscr{G}: d \notin G\}$ as $\{G_n . n \in \omega\}$ and suppose indirectly that for each $n \in \omega$, $D-\bigcup\{G_i : i \leq n\}$ is infinite. Then there is an infinite sequence $y=\{d_n : n \in \omega\} \subset D$ such that for each $n \in \omega$, $d_n \notin \bigcup\{G_i : i \leq n\}$.

Now, pick an arbitrary member x of A Then $x \in G_i$ for some $i \in \omega$ Since G_i is open and $x \in G_i$ it follows that $x-G_i$ is finite. Since $n>i$ implies $d_n \notin G_i$, $x \cap y$ is finite.

By the maximality of A, it follows that $y \in A$ On the other hand, since $y \cap G_n$ is finite for each $n \in \omega$, $y \notin \bigcup\{G_n : n \in \omega\}$, a contradiction to the fact that $\{G_n \; n \in \omega\}$ covers A.

Having proven the claim, let us consider the set

$$D'=\bigcup\{D-\bigcup \mathscr{G}(d): d \in D\}.$$

On the one hand, since there are only countably many finite subsets of $\mathscr{G}$ and each $D-\bigcup \mathscr{G}(d)$ is finite, it follows that D' is countable.

On the other hand, since $d \in D-\bigcup \mathscr{G}(d)$ for each $d \in D$, it follows that $D'=D$, in contradiction with our assumption that D was uncountable. □

3. When S_δ sets or R_δ sets are G_δ sets

Let us start with the following observation.

Theorem 3.1. *If C is a countably compact subspace of a topological space X and C is an S_δ (respectively an R_δ) subset in X, then C is a G_δ (respectively a regular G_δ) subset in X.*

Proof. Let $\mathscr{G}$ be a countable family of open sets which separates (respectively strongly separates) the points of C from the points of $X-C$ Consider

$$\mathscr{G}^* = \{\bigcup \mathscr{G}' \colon \mathscr{G}' \subset \mathscr{G},\ \mathscr{G}' \text{ is a finite cover of } C\}.$$

Then clearly

$$\bigcap \mathscr{G}^* = C \quad (\text{respectively} \bigcap \mathscr{G}^* = \bigcap \{\mathrm{cl}(G^*) \colon G^* \in \mathscr{G}^*\} = C) \qquad \square$$

Corollary 3.1. *Let C be a countably compact subspace in a topological space X. Then the following assertions hold*

(a) *If C is a co-analytic or an open-analytic set in X, then C is a G_δ subset of X.*

(b) (Halmos [11]) *If C is a Baire set in X, then C is a regular G_δ subset of X.*

From Theorem 3 1 it follows that (countably) compact S-perfect spaces are perfect. The following example shows that locally compact S-perfect or even R-perfect, spaces may not be perfect.

Example 3.1 (Gruenhage [10]). There is an R-perfect, locally compact, locally countable space which is not perfect

Proof. Example 2.17 in [10] is a locally compact, locally countable nonperfect refinement of the topology on the real line. This space has a countable strongly separating open cover and, hence, it is R-perfect. $\square$

In the rest of this section it will be shown that S-perfect or R-perfect imply G_δ or regular G_δ in several interesting subclasses of locally compact spaces Several related results will also be given.

Let us begin with the following technical lemma.

Lemma 3.1 (MA(κ)). *Let X be a topological space, and let F be a subset of X such that F is the union of $\leq\kappa$ countably compact subspaces. Then the following implications hold:*

(a) *If F is an S_δ set in X, and $|X-F|\leq\kappa$, then F is a G_δ set in X.*

(b) *If F is an R_δ set in X, and $X-F$ is the union of $\leq\kappa$ countably compact subspaces of X, then F is a G_δ set in X*

(c) *If F is an R_δ set in X, and $X - F$ is the union of $\leqslant \kappa$ countably compact subspaces of X the interiors of which also cover $X - F$, then F is a regular G_δ set in X.*

Remark. Example 3 1 shows that the cardinality restrictions cannot be omitted in any part of Lemma 3.1.

The proofs of (a), (b) and (c) are similar to each other as well as to the proof of the classic theorem that under MA(κ), all subspaces of the reals of cardinality $\leqslant \kappa$ are Q-sets. Actually we only make use of the following well-known consequence of MA(κ).

Lemma 3.2 (MA(κ), see [17, p 57]). *Let $\{A_\alpha \ \alpha \in \kappa\}$ and $\{B_\alpha \cdot \alpha \in \kappa\}$ be two families of subsets of a countable set S such that for all $\alpha \in \kappa$, and for all finite subsets F of κ, $|A_\alpha - \bigcup\{B_\beta \cdot \beta \in F\}| = \omega$ Then there is a subset D of S such that $|A_\alpha \cap D| = \omega$ and $|B_\beta \cap D| < \omega$ for every $\alpha, \beta \in \kappa$.*

Proof of Lemma 3.1. Since the proofs of (a) and (b) are similar to and simpler than that of (c), we shall only prove (c).

So let $\mathcal{F} = \{F_\alpha \ \alpha \in \kappa\}$ and $\mathcal{K} = \{K_\beta : \beta \in \kappa\}$ be two families of countably compact subspaces of X such that $\bigcup \mathcal{F} = F$, $\bigcup \mathcal{K} = X - F$ and $\bigcup\{\mathrm{Int}(K_\beta). \beta \in \kappa\} = X - F$. Also let $\mathcal{G}$ be a countable family of open sets which strongly separates the points of F from the points of $X - F$ Without loss of generality we may assume that $\mathcal{G}$ is closed under finite unions and intersections. Then, by the countably compactness of F_α, for each $\alpha \in \kappa$ and $x \in X - F$, there is a $G \in \mathcal{G}$ with $G \supset F_\alpha$ and $x \notin \mathrm{cl}(G)$. Thus for each $\alpha \in \kappa$ there is decreasing subfamily $\mathcal{A}_\alpha$ of $\mathcal{G}$ such that $F_\alpha \subset \bigcap \mathcal{A}_\alpha$ and $\bigcap\{\mathrm{cl}(G) \ G \in \mathcal{A}_\alpha\} \subset F$. Further, for each $\beta \in \kappa$, let $\mathcal{B}_\beta = \{G \in \mathcal{G}: \mathrm{cl}(G) \cap K_\beta \neq \emptyset\}$. Then, by the countably compactness of K_β, $\mathcal{A}_\alpha - \mathcal{B}_\beta$ is finite for every $\alpha, \beta \in \kappa$ Applying Lemma 3.2 with $\mathcal{A}_\alpha$, $\mathcal{B}_\beta$ and $\mathcal{G}$ in place of A_α, B_β and S, respectively, it follows that there is a subset $\mathcal{D}$ of $\mathcal{G}$ such that $|\mathcal{A}_\alpha \cap \mathcal{D}| = \omega$ and $|\mathcal{B}_\beta \cap \mathcal{D}| < \omega$ for each $\alpha, \beta \in \kappa$. Then $\mathcal{G}^* = \{\bigcup(\mathcal{D} - \mathcal{D}') \ \mathcal{D}'$ is a finite subfamily of $\mathcal{D}\}$ will be a countable family of open sets such that $F = \bigcap \mathcal{G}^* = \bigcap\{\mathrm{cl}(G): G \in \mathcal{G}^*\}$. □

Corollary 3.2 (MA(κ)). *Let F be an S_δ subset of a topological space X with $|X| \leqslant \kappa$ Then F is a G_δ set in X.*

Proof. This is an immediate consequence of Lemma 3.1(a). □

Corollary 3.3 (MA(κ)). *Let F be a closed R_δ set in a regular T_1, locally countably compact, hereditarily $\leqslant\kappa$-Lindelof space X Then F is a regular G_δ set in X*

Proof. This is a consequence of Lemma 3.1(c). □

An application of the above results concerns the following problem of Ross and Stromberg

Problem 3.1 ([22]). Is it true that every closed Baire set is a zero-set in a normal, locally compact T_2 space?

This problem has been answered in the negative (see [2]) We shall see in this paper, however, that the answer is yes in some nice subclasses of locally compact spaces Since Baire sets are R_δ sets and spaces of weight $\leqslant\omega_1$ are $\leqslant\omega_1$-Lindelof, the following result follows from Corollary 3.3.

Corollary 3.4 (MA(ω_1)). *Closed Baire sets are zero-sets in normal, locally compact T_2 spaces of weight ω_1*

Remarks. (1) Note that in Corollary 3 4, normality was only needed to conclude that regular G_δ sets are zero-sets.

(2) MA(ω_1) cannot be omitted from Corollary 3 4 (and thus, MA(κ) cannot be omitted from Corollary 3 3). Indeed, the counterexample to Problem 3.1 constructed in [2] has weight 2^ω which equals ω_1 if CH, an alternative to MA(ω_1), holds

Another nice subclass of normal, locally compact spaces where the answer to Problem 3.1 is yes, is given by the following result of Burke

Theorem 3.2 [6]. *Closed Baire sets are zero-sets in normal, locally compact, submetacompact T_0 spaces.*

Burke's result can be extended to the generality of R_δ sets and S_δ sets and other related results can be proved To do so, we need the following generalization of [16, Theorem 2 18] Note that a set A is called a locally G_δ subset of a space X, if for each $x\in A$, there is an open subset N of X containing x such that $A\cap N$ is a G_δ subset of X

Theorem 3.3. *If F is a closed locally G_δ set in a submetacompact space X, then F is a G_δ set in X*

Proof. Since F is locally G_δ, there is a family $\mathcal{N}=\{N(\alpha).\ \alpha\in\lambda\}$ of open subsets of X such that $\mathcal{N}$ covers F and for every $\alpha\in\lambda$,

(a) $P(\alpha)=(N(\alpha)\cap F)-\bigcup\{N(\beta)\ \ \beta\in\alpha\}\neq\emptyset$,

(b) there is a countable decreasing family $\{G(\alpha,n)\ \ n\in\omega\}$ of open subsets of $N(\alpha)$ with $\bigcap\{G(\alpha,n): n\in\omega\}=F\cap N(\alpha)$.

For each subset L of X, let $\alpha(L)$ denote the smallest ordinal $\alpha\leqslant\lambda$ such that $L\subset(X-F)\cup(\bigcup\{P_\delta\ \ \delta\leqslant\alpha\})$ Also, for each open cover $\mathcal{U}$ of X, and for each $x\in X-F$, let $\beta(x,\mathcal{U})=\alpha(\mathrm{st}(x,U))$

Claim For each open cover $\mathcal{U}$ of X, there is a countable family $c(\mathcal{U})$ of open covers such that for each $x\in X-F$ with $\mathrm{st}(x,\mathcal{U})\cap F\neq\emptyset$, there is a $\mathcal{V}\in c(\mathcal{U})$ with $\beta(x,\mathcal{V})<\beta(x,\mathcal{U})$

In order to formulate the proof of the claim, let us set $P(\lambda)=\emptyset$ and $G(\lambda, n)=\emptyset$ for each $n\in\omega$ Now, for each $\alpha\leqslant\lambda$ and $n\in\lambda$ let

$$U(\alpha, n)=G(\alpha, n)\cap(\bigcup\{U\in\mathscr{U}\ \alpha(U)\geqslant\alpha\}).$$

For every $n\in\omega$, $\mathscr{U}(n)=\{X-F\}\cup\{U(\alpha, n): \alpha\leqslant\lambda\}$ is an open cover of X, and thus it has a θ-sequence $\{\mathscr{V}(n, i)\ i\in\omega\}$ of open refinements We are going to show that $c(\mathscr{U})=\{\mathscr{V}(n, i). n, i\in\omega\}$ is as required.

To see this, let x be an arbitrary point of $X-F$ with $\mathrm{st}(x, \mathscr{U})\cap F\neq\emptyset$, and let $\beta=\beta(x, \mathscr{U})$ Since $x\notin F$ and $\bigcap\{G(\beta, n): n\in\omega\}\subset F$, there is an $n\in\omega$ with $x\notin G(\beta, n)$ By this and the definition of β, $x\notin\bigcup\{U(\alpha, n). \alpha\geqslant\beta\}$. Let $i\in\omega$ be such that $\mathscr{V}(n, i)$ is point-finite at x Since $\mathscr{V}(n, i)$ refines $\mathscr{U}(n)$, for every V with $x\in V\in\mathscr{V}(n, i)$ V is contained in a member of $\{X-F\}\cup\{U(\alpha, n)\cdot \alpha<\beta\}$ Since $U(\alpha, n)\subset N(\alpha)$ it follows that $\beta(x, \mathscr{V}(n, i))<\beta(x, \mathscr{U})$.

Having proven the claim, let us inductively define a sequence $\mathscr{C}(n)$ $(n\in\omega)$ of countable families of open covers of X as follows

(1) $\mathscr{C}(0)$ consists of the single open cover $\{X-F\}\cup\{N(\alpha)\ \alpha\in\lambda\}$

(2) If $\mathscr{C}(n)$ is already defined, then $\mathscr{C}(n+1)=\bigcup\{c(\mathscr{U}). \mathscr{U}\in\mathscr{C}(n)\}$

Finally let a countable family $\mathscr{C}$ of open covers of X be defined by $\mathscr{C}=\bigcup\{\mathscr{C}(n)\cdot n\in\omega\}$. Since there is no strictly decreasing infinite sequence of ordinals, by the claim and the construction of $\mathscr{C}$ it follows that

$$\text{for each } x\in X-F, \text{ there is } \mathscr{U}\in\mathscr{C} \text{ such that } \mathrm{st}(x, \mathscr{U})\cap F=\emptyset \tag{5}$$

Thus $F=\bigcap\{\mathrm{st}(F, U)\ U\in\mathscr{C}\}$ is indeed a G_δ set in X □

Remark. Theorem 3 3 seems to be new even in the class of paracompact spaces rather than submetacompact spaces although the proof is easier for paracompact spaces.

Theorem 3.4. *A subset F is a G_δ subset of a submetacompact space X if one of the following conditions holds·*

(i) *F is a closed S_δ subset of X, and X is locally (countably) compact*

(ii) *(MA(κ))F is an S_δ subset of X and X is locally of cardinality $\leqslant\kappa$.*

Proof. (i) and (ii) follow from Theorem 3 3, Theorem 3 1 and Lemma 3.1(a) respectively □

Corollary 3.5. *Closed S_δ sets are zero-sets in normal, locally compact, submetacompact T_0 spaces.*

Proof. This corollary immediately follows from Theorem 3 4(i). □

4. Mappings of R-perfect and S-perfect spaces

Definition 4.1. A continuous function f $X \to Y$ is said to be *closed* if the images of closed sets are closed. If, in addition to being closed, the inverse images of points are compact, then f is a *perfect* map. If f takes open sets to open sets and the inverse images of points are compact, then f is an *open-compact* map

The next example shows that neither R-perfectness nor S-perfectness of spaces need be preserved under closed mappings.

Example 4.1. Let M denote the Michael line [19] and Y the quotient space obtained by mapping all rationals to a point p Clearly the quotient map f is closed M is R-perfect (see Theorem 2.2(iii)). Since Q, the set of rationals in the real line, is not a G_δ set in the Michael line, the singleton set $\{p\}$ cannot be a G_δ set in Y Since singletons are S_δ sets or R_δ sets if and only if they are G_δ sets, Y is neither S-perfect nor R-perfect

The next several theorems show that R-perfect spaces and S-perfect spaces are preserved under perfect maps

Theorem 4.1. *The perfect image of an S-perfect space is S-perfect.*

Proof. Let f be a perfect map from an S-perfect space X onto Y. If F is closed in Y let $\mathcal{G}$ be a countable collection of open subsets of X that separates the points of $f^{-1}(F)$ from those of $X-f^{-1}(F)$ No generality is lost if it is assumed that $\mathcal{G}$ is closed under finite unions. If $f^{-1}(x) \subseteq f^{-1}(F)$ and $z \notin f^{-1}(F)$, then for each $t \in f^{-1}(x)$ there exists $G \in \mathcal{G}$ such that $z \notin G$ and $t \in G$ Since $f^{-1}(x)$ is compact and $\mathcal{G}$ is closed under finite unions there exists a $G_x \in \mathcal{G}$ such that $f^{-1}(x) \subset G_x$ and $z \notin G_x$

Let $K(G) = Y - f(X-G)$ for each $G \in \mathcal{G}$, and let $\mathcal{K} = \{K(G)\ \ G \in \mathcal{G}\}$ It will be shown that $\mathcal{K}$ separates the points of F from those of $Y-F$. Indeed if $x \in F$ and $y \notin F$, then there exists $G \in \mathcal{G}$ such that $f^{-1}(x) \subset G$ and $z \notin G$ for some $z \in f^{-1}(y)$ Since $f^{-1}(x) \subset G$, $x \in K(G)$. If y was in $K(G)$, then it would follow that $y \notin f(X-G)$ Hence $f^{-1}(y) \cap (X-G) = \emptyset$, in contradiction with our assumption Hence $y \notin K(G)$ and, thus, $\mathcal{K}$ separates the points of F from $Y-F$ □

Theorem 4.2. *The perfect image of an R-perfect space is R-perfect.*

Proof. Let f be a perfect map from an R-perfect space X onto a space Y If F is closed in Y, then let $\mathcal{G}$ be a countable collection of open subsets of X that strongly separates the points of $f^{-1}(F)$ from the points of $X-f^{-1}(F)$. Without loss of generality assume $\mathcal{G}$ is closed under finite unions and finite intersections Then given $f^{-1}(x) \subset f^{-1}(F)$ and $z \notin f^{-1}(F)$ there exists $G \in \mathcal{G}$ such that $f^{-1}(x) \subset G$ and $z \notin \mathrm{cl}(G)$ Let $G_1, G_2, \ldots$ be a listing of the elements of $\mathcal{G}$ that contain $f^{-1}(x)$.

Let $f^{-1}(y) \cap f^{-1}(F) = \emptyset$ and suppose for each $n \in \omega$,

$$F_n = \mathrm{cl}(G_1) \cap \mathrm{cl}(G_2) \cap \cdots \cap \mathrm{cl}(G_n) \cap f^{-1}(y) \neq \emptyset.$$

Then $\bigcap\{F_n\ n\in\omega\}\neq\emptyset$ since $f^{-1}(y)$ is compact Let $z\in\bigcap\{F_n . n\in\omega\}$. There exists G_i such that $z\notin \mathrm{cl}(G_i)$, a contradiction. Thus there exists an $n\in\omega$ such that $f^{-1}(y)\cap \mathrm{cl}(G_n)=\emptyset$

For each $G\in\mathscr{G}$, $X-G$ is closed and, since f is a closed map, $f(X-G)$ is closed. Hence $H(G)=Y-f(X-G)$ is open and $\mathscr{H}=\{H(G)\cdot G\in\mathscr{G}\}$ is a countable collection of open subsets of Y Notice that $\mathrm{cl}(H(G))\subseteq f(\mathrm{cl}(G))$.

Let $x\in F$ and $y\notin F$. There exists $G\in\mathscr{G}$ such that $f^{-1}(x)\subset G$ and $\mathrm{cl}(G)\cap f^{-1}(y)=\emptyset$ It follows that $x\in H(G)$ and $y\notin f(\mathrm{cl}(G))$ Thus $y\notin \mathrm{cl}(H(G))$ and $\mathscr{H}$ separates the points of F from the points of $X-F$ □

The next two examples show that neither R-perfect spaces nor S-perfect spaces need to be preserved under open-compact mappings These examples have implications in the class MOBI (see [1] or [7]).

Example 4.2. Let $\mathrm{PR}(X)$ be the metacompact Moore space in Example 2 1 that is not R-perfect Since metacompact Moore spaces are open-compact images of metric spaces [12] and since metric spaces are perfectly normal (hence R-perfect) it follows that the perfect images of an R-perfect space need not be R-perfect

The following construction of Bing is found in [7].

Let Z be a completely regular first countable space such that the set A of accumulation points of Z is discrete Let

$$Z^*=\{(z,\psi).\ z\in Z-A,\ \psi \text{ is a countably infinite subset of } A\}.$$

Let $Y=A\cup Z^*$ be topologized so that each point of Z^* is discrete and, if U is an open set in Z containing $a\in A$, then

$$\{a\}\cup\{(z,\psi)\in Z^*\ \ a\in\psi \text{ and } z\in U\}=[a,U]$$

forms a local base for a Then Y is completely regular and is the open-compact image of a metacompact Moore space Notice A is a closed discrete subset of Y

Lemma 4.1. *Using the above notation A is S_δ in Z if and only if A is S_δ in Y*

Proof. Let $\mathscr{G}$ be a countable collection of open subsets of Z that separates the points of A from the points of $Z-A$. For each $G\in\mathscr{G}$ let

$$H(G)=\bigcup\{[a,G]\colon a\in A\cap G\}$$

Notice that $H(G)$ is open and let $\mathscr{H}=\{H(G)\ \ G\in\mathscr{G}\}$ Let $a\in A$ and $(a,\psi)\in Z^*$ Choose $G\in\mathscr{G}$ such that $a\in G$ and $z\notin G$ Then $a\in H(G)$ and $(z,\psi)\notin H(G)$. Thus $\mathscr{H}$ separates A from $Y-A$

Conversely let $\mathscr{G}$ be a countable collection of open sets that separates the points of A from the points of $Y-A=Z^*$ in Y. For each $G\in\mathscr{G}$ and $a\in G\cap A$ choose an open set $V(a,G)$ in Z containing a such that $[a,V(a,G)]\subset G$. Let $G'=\bigcup\{[a,V(a,G)].\ a\in G\cap A\}$ Then $\mathscr{G}'=\{G'\ \ G\in\mathscr{G}\}$ also separates the points of A from the points of Z^* Let $H(G')=\bigcup\{V(a,G)\cdot a\in G\cap A\}$ for each $G'\in\mathscr{G}'$ and let $\mathscr{H}=\{H(G')\cdot G'\in\mathscr{G}'\}$

Suppose there exists $a \in A$ and $z \in Z - A$ such that for every $H \in \mathscr{H}$, if $a \in H$, then $z \in H$. Let $H_1, H_2, \ldots$ list all $H \in \mathscr{H}$ such that $a \in H$ Then, for each $n \in \omega$, there exists $a_n \in A$ such that $z \in V(a_n, G_n)$ where $H_n = H(G'_n)$ Let $\psi = \{a_n \mid n \in \omega\}$ Then a is not separated from (z, ψ) by $\mathscr{G}'$, a contradiction. Thus $\mathscr{H}$ separates the points of A from the points of $Z - A$ in Z. □

Example 4.3. Let Z be the space of Example 2.3 Then Z satisfies the conditions of the construction preceding Lemma 4 1. Let Y be the space obtained from this construction. Since Z in not S-perfect, Y is not S-perfect by Lemma 4 1. But Y is the open-compact image of a metacompact Moore space which is S-perfect. Hence the open-compact image of an S-perfect space need not be S-perfect.

Using Examples 4 2 and 4 3 it follows that spaces in MOBI need not be either R-perfect or S-perfect.

The following weak preservation theorem does hold for open-compact maps

Theorem 4.3. *The open-compact image of an R-perfect space is S-perfect.*

Proof. Let f be an open-compact map from the R-perfect space X onto the space Y. Let F be a closed subset of Y and $\mathscr{G}$ a countable collection of open subsets of X that strongly separates the points of $f^{-1}(F)$ from the points of $X - f^{-1}(F)$ Without loss of generality let $\mathscr{G}$ be closed under finite unions and finite intersections. Thus, as in Theorem 4.2, if $f^{-1}(x) \subset f^{-1}(F)$ and $f^{-1}(y) \cap f^{-1}(F) = \emptyset$, then there is a $G \in \mathscr{G}$ such that $f^{-1}(x) \subset G$ and $\mathrm{cl}(G) \cap f^{-1}(y) = \emptyset$

For each $G \in \mathscr{G}$ let $H(G) = \{t \in Y \mid f^{-1}(t) \cap G \neq \emptyset\}$. Then $\mathscr{H} = \{H(G) \mid G \in \mathscr{G}\}$ is a countable collection of open subsets of Y that separates the points of F from the points of $Y - F$ To see this, let $x \in F$ and $y \in Y - F$. Then there exists $G \in \mathscr{G}$ such that $f^{-1}(x) \subset G$ and $\mathrm{cl}(G) \cap f^{-1}(y) = \emptyset$ and so $G \cap f^{-1}(y) = \emptyset$. Hence $x \in H(G)$ If y was in $H(G)$, then $f^{-1}(y) \cap G \neq \emptyset$ would hold. Thus $y \notin H(G)$ and $\mathscr{H}$ separates the points of F from the points of $Y - F$. □

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